NUMERICAL SOLUTION OF TIME-FRACTIONAL CAUCHY REACTION-DIFFUSION EQUATION USING ABOODH TRANSFORM METHOD

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ABSTRACT

This study focused on numerical solution of time-fractional Cauchy reaction-diffusion equation by Aboodh Transform Method. Three examples were considered to demonstrate the effectiveness of the proposed method. The solution resulting from the proposed method are presented in series. Comparing the results obtained with other existing methods, it was observed that the results are exactly equal. The result indicating that the Aboodh Transform method is well suitable for solving such and related problems.

Keywords: Aboodh transform, Caputo fractional derivative, Cauchy-diffusion equation. Diffusion equation, Time-Fractional Cauchy Reaction-Diffusion Equations.

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INTRODUCTION

Fractional differential calculus which is a generalization of differential calculus have received much interest in Physics, Mathematics and Applied sciences [1-7]. Several physical phenomena can be modeled by applying the theory of fractional calculus. Recently, the fractional calculus has a wide range of application in the mathematical modeling of real world. Recently, fractional differential equations played a major role in diverse areas such as Viscoelasticity, Biology, Physics, Engineering and many other applications.

However, several fractional calculus does not possess exact analytical solutions and thus numerical methods are developed and applied in solving them. Several numerical techniques have been applied to obtain solutions to fractional calculus. Some of those numerical techniques are: Adomian Decomposition Method [8], Variational Iteration Method [9, 10], Optimal Homotopy Asymptotic Method [4], Homotopy Analysis Method [4, 11]. Time –fractional Cauchy reaction-diffusion equations [12, 13] is one of the major class of fractional partial differential equations. The time–fractional Cauchy reaction- Cauchy-reaction diffusion equations specify many forms of nonlinear systems in physical sciences, biological sciences and engineering sciences [14, 15]. The numerical solutions of Cauchy reaction-diffusion equation had been received by means of the usage of several numerical approaches, namely, Homotopy perturbation Method [11, 16], Generalized differential transform and finite difference methods [17], Sumudu Iterative Method [18]. The primary aim of this paper is to establish Aboodh Transform Method (ATM) for the numerical solution of the time-fractional Cauchy reaction diffusion equation. This method is a modification of the work of Aboodh [19]. This method obtained its result in a series form which converges rapidly.

Definition of basic terms

Here, the primary definitions and features of fractional calculus and Aboodh transform were given to be used in this work.

Definition 1

A real function $f(x)$, $x > 0$, is said to be in space $C_{\mu,}\ \mu \epsilon \mathbb{R}$, if there exists a real number p, $(p > \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$ and it is said to be in space C_μ^m if $f^{(m)} \in C_{\mu,}\ m \in \mathbb{N}$. [22].

Definition 2

The Riemann-Liouville fractional integral operator of order $\propto \geq 0$ of a function $f(x) \in C_\mu, \mu \geq -1$ is defined as [11]

$$I^{\propto} f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \alpha > 0, x > 0 \\ I^0 f(x) = f(x),\ \alpha = 0 \end{cases} \tag{1}$$

Where $\Gamma(.)$ is the Gamma function

The following features of operator I^α, which will be used in the context of this paper are as follow. For $f \in C_{\mu,}\ \mu, \gamma = -1, \alpha, \beta \geq 0$.

$$I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x) = I^{\alpha+\beta} f(x) \tag{2}$$

$$I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \tag{3}$$

Definition 3

The fractional derivative of $f(x)$ in the Caputo sense is defined as [11]

$$D^\alpha f(x) = I^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^\alpha (x-t)^{n-\alpha-1} f^{(n)}(t)dt \tag{4}$$

Definition 4

The Aboodh Transform defined over the set of function $A = \{f(t): \exists\, M, k_1, k_2 > 0, |f(t)| < Me^{-vt}\}$ is represented by the formular [19]

$$A[f(t)] = k(v) = \frac{1}{v}\int_0^\infty f(t)\, e^{-vt} dt, t \geq 0, k_1 \leq v \leq k_2 \tag{5}$$

Definition 5

The Mittag-Leffler function E_α with $\alpha > 0$ is defined as [10]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^\alpha}{\Gamma(n\alpha+1)} \tag{6}$$

Definition 6

The Aboodh Transform of the Caputo fractional derivative is defined as

$$A[D_t^{n\alpha} u(x,t)] = v^{n\alpha} Au[(x,t)] - \sum_{k=0}^{n-1} \frac{u^{(k)}(x,0)}{v^{2n\alpha+k}}$$

for $n-1 < \alpha < n, n = 1, 2, 3, \cdots$

MATERIALS AND METHODS

Aboodh transform method

To illustrate the basic idea of Aboodh Transform Method (ATM), there is need to consider the following equation with the prescribed initial condition as

$$D_t^{n\alpha} u(x,t) + Lu(x,t) + Ru(x,t) = g(x,t), \quad n-1 < n\alpha \leq n, \tag{7}$$

$u(x,0) = h(x)$.

Where $D_t^{n\alpha} u(x,t)$ is the Caputo fractional derivative operator, $D_t^{n\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, L is a linear differential operator, R is the general nonlinear differential operator, $g(x,t)$ is the homogeneous term.

Applying Aboodh Transform on both sides of equation (7), we have

$$A[D_t^{n\alpha} u(x,t)] + A[Lu(x,t) + Ru(x,t)] = A[g(x,t)] \tag{8}$$

Applying the differential properties of the Aboodh Transform, we have

$$A[u(x,t)] - v^{n\alpha} \sum_{k=0}^{n-1} u^k(x,0) + v^{n\alpha} A[Lu(x,t) + Ru(x,t) - g(x,t) = 0] \tag{9}$$

$$A[u(x,t)] = v^{n\alpha} \sum_{k=0}^{n-1} u^k(x,0) - v^{n\alpha} A[Lu(x,t) + Ru(x,t) - g(x,t)] \tag{10}$$

Taking the Aboodh inverse on both sides of equation (10)

$$u(x,t) = A^{-1}[v^{n\alpha} \sum_{k=0}^{n-1} u^k(x,0)] - A^{-1}\left[v^{n\alpha} A[Lu(x,t) + Ru(x,t) - g(x,t)]\right] \tag{11}$$

Let assuming the following parameters:

$$f(x,t) = A^{-1}\left[v^{n\alpha} \sum_{k=0}^{n-1} u^k(x,0) + v^\alpha A[g(x,t)]\right]$$

$$N(u(x,t)) = -A^{-1}\left[v^{n\alpha} A[Ru(x,t)]\right]$$

$$K(u(x,t)) = -A^{-1}\left[v^{n\alpha} A[Lu(x,t)]\right]$$

Therefore, equation (11) can now be written as

$$u(x,t) = f(x,t) + K(u(x,t)) + N(u(x,t)), \tag{12}$$

Where f is a known function, K and N are given linear and nonlinear term of u, respectively.

The solution of Equation (12) can be written in the series form

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \tag{13}$$

We have

$$K(\sum_{i=0}^{\infty} u_i) = \sum_{i=0}^{\infty} K(u_i) \tag{14}$$

The nonlinear term N can be decomposed as

$$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=0}^{\infty}\{N(\sum_{j=0}^{i} u_j) - (\sum_{j=0}^{i} u_j)\} \tag{15}$$

Thus, equation (12) can be written in the following form

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} K(u_i) + N(u_0) + \sum_{i=0}^{\infty}\{N(\sum_{j=0}^{i} u_j) - (\sum_{j=0}^{i} u_j)\} \tag{16}$$

Defining the recurrence relation

$$u_0 = f$$
$$u_1 = K(u_0) + N(u_0),$$
$$u_{r+1} = K(u_r) + N(u_0 + u_1 + \cdots + u_r) - N(u_0 + u_1 + \cdots + u_{r-1}) \tag{17}$$

We have

$$(u_1 + \cdots + u_{r+1}) = K(u_0 + u_1 + \cdots + u_r) + N(u_0 + u_1 + \cdots + u_r) \tag{18}$$

Thus,

$$\sum_{i=0}^{\infty} u_i = f + K(\sum_{i=0}^{\infty} u_i) + N(\sum_{i=0}^{\infty} u_i) \tag{19}$$

The m-term approximate solution of equation (12) is written as

$$u = u_0 + u_1 + \cdots + u_{m-1} \tag{20}$$

RESULTS AND DISCUSSION

Here, in view of showing the validity and applicability of the Aboodh Transform Method on Time-Fractional Cauchy Reaction-Diffusion Equations, the following examples will be considered.

Example 1

Examine the time-fractional Cauchy reaction-diffusion equation [18]

$$u_t^{\alpha}(x,t) = u_{xx}(x,t) - u(x,t), \ 0 < \alpha \leq 1 \tag{21}$$

subject to initial conditions

$$u(x,0) = e^{-x} + x$$

The exact solution of equation (21) is $e^{-x} + xe^{-t}$.

Taking the Aboodh Transform of equation (21) we have,

$$A[u_t^{\propto}(x,t)] = A[u_{xx}(x,t) - u(x,t)] \qquad (22)$$

Applying the differential property of the Aboodh Transform

$$v^{\propto}u(x,v) - \frac{u(x,0)}{v^{2-\propto}} = A[u_{xx}(x,t) - u(x,t)] \qquad (23)$$

$$u(x,v) = \frac{u(x,0)}{v^2} + \frac{1}{v^{\propto}}A[u_{xx}(x,t) - u(x,t)] \qquad (24)$$

Taking the inverse Aboodh Transform on both sides of equation (24)

$$u(x,t) = A^{-1}\left\{\frac{u(x,0)}{v^2}\right\} + A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{xx}(x,t) - u(x,t)]\right\} \qquad (25)$$

Substituting the initial value $(x,0) = e^{-x} + x$

$$u(x,t) = A^{-1}\left\{\frac{u(x,0)}{v^2}\right\} + A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{xx}(x,t) - u(x,t)]\right\} \qquad (26)$$

$$u(x,t) = e^{-x} + x + A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{xx}(x,t) - u(x,t)]\right\} \qquad (27)$$

$$u_0(x,t) = e^{-x} + x, \quad u_n(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{n-1xx}(x,t) - u_{n-1}(x,t)]\right\} \qquad (28)$$

Other values of $u(x,t)$ can be obtained by applying the successive iteration

$$u_n(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{n-1xx}(x,t) - u_{n-1}(x,t)]\right\} \qquad (29)$$

when $n = 1$, $u_1(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{0xx}(x,t) - u_0(x,t)]\right\}$

$$u_1(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[e^{-x} - (e^{-x} + x)]\right\}$$

$$u_1(x,t) = \frac{-xt^{\propto}}{\Gamma(\propto + 1)}$$

when $n = 2$, $u_2(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{1xx}(x,t) - u_1(x,t)]\right\}$

$$u_2(x,t) = \frac{xt^{2\propto}}{\Gamma(2\propto + 1)}$$

when $n = 3$, $u_3(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{2xx}(x,t) - u_2(x,t)]\right\}$

$$u_3(x,t) = \frac{-xt^{3\propto}}{\Gamma(3\propto + 1)}$$

when $n = 4$, $u_4(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{3xx}(x,t) - u_3(x,t)]\right\}$

$$u_4(x,t) = \frac{xt^{4\propto}}{\Gamma(4\propto + 1)}$$

$\vdots$

$$u_n(x,t) = (-1)^n\frac{xt^{n\propto}}{\Gamma(n\propto + 1)}$$

Other terms of the iteration can be obtained by following the same principle.

The approximate solution of equation (21) can be written as:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots + u_n$$

$$u(x,t) = e^{-x} + x - \frac{xt^{\propto}}{\Gamma(\propto+1)} + \frac{xt^{2\propto}}{\Gamma(2\propto+1)} - \frac{xt^{3\propto}}{\Gamma(3\propto+1)} + \frac{xt^{4\propto}}{\Gamma(4\propto+1)} - \frac{xt^{5\propto}}{\Gamma(5\propto+1)} + \cdots + (-1)^n\frac{xt^{n\propto}}{\Gamma(n\propto+1)} \qquad (30)$$

$$u(x,t) = e^{-x} + x\left(1 - \frac{t^{\propto}}{\Gamma(\propto+1)} + \frac{t^{2\propto}}{\Gamma(2\propto+1)} - \frac{t^{3\propto}}{\Gamma(3\propto+1)} + \frac{t^{4\propto}}{\Gamma(4\propto+1)} - \frac{t^{5\propto}}{\Gamma(5\propto+1)} + \cdots + (-1)^n\frac{t^{n\propto}}{\Gamma(n\propto+1)}\right) \qquad (31)$$

The solution can be written in the form

$$u(x,t) = e^{-x} + xE_{\propto}(-t^{\propto}) \qquad (32)$$

The equation (32) is the approximate solution to equation (21). when $\propto = 1$ equation (32) becomes $u(x,t) = e^{-x} + xe^{-t}$ which is the exact solution for equation (21).

The result obtained agrees with the solution in [18].

Example 2

Considering the nonlinear time-fractional Cauchy reaction-diffusion equation [18]

$$u_t^{\propto}(x,t) = u_{xx} - u_x + uu_{xx} - u^2 + u, \qquad 0 < \propto \leq 1 \qquad (33)$$

$$u(x,0) = e^x$$

$u(x,t) = e^{x+t}$ is the exact solution of equation (33)

Applying the Aboodh Transform of equation (33) yields

$$A[u_t^{\propto}] = A[u_{xx} - u_x + uu_{xx} - u^2 + u]$$

(34)

Applying the differential property of the Aboodh Transform

$$v^{\propto}u(x,v) - \frac{u(x,0)}{v^{2-\propto}} = A[u_{xx} - u_x + uu_{xx} - u^2 + u] \quad (35)$$

$$u(x,v) = \frac{u(x,0)}{v^2} + \frac{1}{v^{\propto}}A[u_{xx} - u_x + uu_{xx} - u^2 + u] \quad (36)$$

Taking inverse Aboodh Transform on both sides of equation (36)

$$u(x,t) = A^{-1}\left\{\frac{u(x,0)}{v^2}\right\} + A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{xx} - u_x + uu_{xx} - u^2 + u]\right\} \quad (37)$$

Substituting the initial value $(x,0) = e^x$

$$u(x,t) = A^{-1}\left\{\frac{u(x,0)}{v^2}\right\} + A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{xx}(x,t) - u(x,t)]\right\} \quad (38)$$

$$u_0(x,t) = e^x,$$

$$u_n(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{n-1xx} - u_{n-1x} + u_{n-1}u_{n-1xx} - u^2{}_{n-1} + u_{n-1}]\right\}$$

Other values of $u(x,t)$ can be obtained by applying the successive iteration

$$u_n(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{n-1xx} - u_{n-1x} + u_{n-1}u_{n-1xx} - u^2{}_{n-1} + u_{n-1}]\right\}$$

(39)

when $n = 1$, $u_1(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{0xx} - u_{0x} + u_0u_{0xx} - u^2{}_0 + u_0]\right\}$

$$u_1(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[e^x - e^x + e^xe^x - e^{2x} + e^x]\right\}$$

$$u_1(x,t) = \frac{e^xt^{\propto}}{\Gamma(\propto +1)}$$

when $n = 2$, $u_2(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{1xx} - u_{1x} + u_1u_{1xx} - u^2{}_1 + u_1]\right\}$

$$u_2(x,t) = \frac{e^xt^{2\propto}}{\Gamma(2\propto +1)}$$

when $n = 3$, $u_3(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{2xx} - u_{2x} + u_2u_{2xx} - u^2{}_2 + u_2]\right\}$

$$u_3(x,t) = \frac{e^xt^{3\propto}}{\Gamma(3\propto +1)}$$

when $n = 4$, $u_4(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{3xx} - u_{3x} + u_3u_{3xx} - u^2{}_3 + u_3]\right\}$

$$u_4(x,t) = \frac{e^xt^{4\propto}}{\Gamma(4\propto +1)}$$

when $n = 5$, $u_4(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{4xx} - u_{4x} + u_4u_{4xx} - u^2{}_4 + u_4]\right\}$

$$u_5(x,t) = \frac{e^xt^{5\propto}}{\Gamma(5\propto +1)}$$

$$u_n(x,t) = \frac{e^xt^{n\propto}}{\Gamma(n\propto +1)}$$

Other terms of the iteration can be obtained by following the same principle.

The approximate solution of equation (33) can be written as:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots + u_n$$

$$u(x,t) = e^x + \frac{e^xt^{\propto}}{\Gamma(\propto+1)} + \frac{e^xt^{2\propto}}{\Gamma(2\propto+1)} + \frac{e^xt^{3\propto}}{\Gamma(3\propto+1)} + \frac{e^xt^{4\propto}}{\Gamma(4\propto+1)} + \frac{e^xt^{5\propto}}{\Gamma(5\propto+1)} + \cdots + \frac{e^xt^{n\propto}}{\Gamma(n\propto+1)} \quad (40)$$

$$u(x,t) = e^x\left[1 + \frac{t^{\propto}}{\Gamma(\propto+1)} + \frac{t^{2\propto}}{\Gamma(2\propto+1)} + \frac{t^{3\propto}}{\Gamma(3\propto+1)} + \frac{t^{4\propto}}{\Gamma(4\propto+1)} + \frac{t^{5\propto}}{\Gamma(5\propto+1)} + \cdots + \frac{t^{n\propto}}{\Gamma(n\propto+1)}\right] \quad (41)$$

The resulting solution can be written in the form

$$u(x,t) = e^xE_{\propto}(t^{\propto})$$

(42)

The equation (42) is the approximate solution to equation (33). when $\propto = 1$ equation (42) becomes $u(x,t) = e^{x+t}$ which is the exact solution of equation (33).

The result obtained agrees with the solution in [18].

Example 3

Considering the time-fractional Cauchy reaction-diffusion equation [18]

$$u_t^{\propto}(x,t) = u_{xx}(x,t) - (1+4x^2)u(x,t), \quad 0 < \propto \leq 1 \quad (43)$$

$$u(x,0) = e^{x^2}$$

The exact solution of equation (43) is $u(x,t) = e^{x^2} + t$.

Taking the Aboodh Transform of equation (43) we have,

$$A[u_t^{\propto}(x,t)] = A[u_{xx}(x,t) - (1+4x^2)u(x,t)] \quad (44)$$

Applying the differential property of the Aboodh Transform

$$v^{\propto}u(x,v) - \frac{u(x,0)}{v^{2-\propto}} = A[u_{xx}(x,t) - (1+4x^2)u(x,t)] \quad (45)$$

$$u(x,v) = \frac{u(x,0)}{v^2} + \frac{1}{v^{\propto}}A[u_{xx}(x,t) - (1+4x^2)u(x,t)] \quad (46)$$

Taking the inverse Aboodh Transform on each sides of equation (46)

$$u(x,t) = A^{-1}\left\{\frac{u(x,0)}{v^2}\right\} + A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{xx}(x,t) - (1+4x^2)u(x,t)]\right\} \quad (47)$$

Substituting the initial value $u(x,0) = e^{x^2}$, we have

$$u(x,t) = e^{x^2} + A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{xx}(x,t) - (1+4x^2)u(x,t)]\right\} \quad (48)$$

$$u_0(x,t) = e^{x^2}, \quad u_n(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{n-1xx}(x,t) - (1+4x^2)u_{n-1}(x,t)]\right\} \quad (49)$$

Other values of $u(x,t)$ can be obtained by applying the successive iteration

$$u_n(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{n-1xx}(x,t) - (1+4x^2)u_{n-1}(x,t)]\right\} \quad (50)$$

when $n = 1$, $u_1(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{0xx}(x,t) - (1+4x^2)u_0(x,t)]\right\}$

$$u_1(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A\left[2e^{x^2} + 4x^2e^{x^2} - (1+4x^2)(e^{-x}+x)\right]\right\}$$

$$u_1(x,t) = \frac{e^{x^2}t^{\propto}}{\Gamma(\propto+1)}$$

when $n = 2$, $u_2(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{1xx}(x,t) - (1+4x^2)u_1(x,t)]\right\}$

$$u_2(x,t) = \frac{e^{x^2}t^{2\propto}}{\Gamma(2\propto+1)}$$

when $n = 3$, $u_3(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{2xx}(x,t) - (1+4x^2)u_2(x,t)]\right\}$

$$u_3(x,t) = \frac{e^{x^2}t^{3\propto}}{\Gamma(3\propto+1)}$$

when $n = 4$, $u_4(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{3xx}(x,t) - (1+4x^2)u_3(x,t)]\right\}$

$$u_4(x,t) = \frac{e^{x^2}t^{4\propto}}{\Gamma(4\propto+1)}$$

when $n = 5$, $u_4(x,t) = A^{-1}\left\{\frac{1}{v^{\propto}}A[u_{4xx}(x,t) - u_4(x,t)]\right\}$

$$u_5(x,t) = \frac{e^{x^2}t^{5\propto}}{\Gamma(5\propto+1)}$$

$$\vdots$$

$$u_n(x,t) = \frac{e^{x^2}t^{n\propto}}{\Gamma(n\propto+1)}$$

Other terms of the iteration can be obtained by following the same principle.

The approximate solution of equation (43) can be written as:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots + u_n$$

$$u(x,t) = e^{x^2} + \frac{e^{x^2}t^{\propto}}{\Gamma(\propto+1)} + \frac{e^{x^2}t^{2\propto}}{\Gamma(2\propto+1)} + \frac{e^{x^2}t^{3\propto}}{\Gamma(3\propto+1)} + \frac{e^{x^2}t^{4\propto}}{\Gamma(4\propto+1)} + \frac{e^{x^2}t^{5\propto}}{\Gamma(5\propto+1)} + \cdots + \frac{e^{x^2}t^{n\propto}}{\Gamma(n\propto+1)} \quad (51)$$

$$u(x,t) = e^{x^2}\left(1 + \frac{t^{\propto}}{\Gamma(\propto+1)} + \frac{t^{2\propto}}{\Gamma(2\propto+1)} + \frac{t^{3\propto}}{\Gamma(3\propto+1)} + \frac{t^{4\propto}}{\Gamma(4\propto+1)} + \frac{t^{5\propto}}{\Gamma(5\propto+1)} + \cdots + \frac{t^{n\propto}}{\Gamma(n\propto+1)}\right) \quad (52)$$

The result can be written in the form

$$u(x,t) = e^{x^2}E_{\propto}(t^{\propto}) \quad (53)$$

The equation (53) is the approximate solution to equation (43).

When $\propto = 1$ equation (43) becomes:

$$u(x,t) = e^{x^2+t} \quad (54)$$

Equation (54) is the exact solution for equation (43).

The result obtained agrees with the solution in [18].

CONCLUSION

Aboodh Transform Method (ATM) has been applied in this paper, to obtain the approximate result for the time-fractional Cauchy Reaction–Diffusion Equations. The major benefit of this technique is the ability to give the solution in series of sequence which converges rapidly. The results obtained show that the Aboodh Transform Method is trustworthy and introduces a significant advancement in solving partial differential equations over existing methods.

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